

ASYMPTOTIC GROWTH OF SATURATED POWERS AND EPSILON MULTIPLICITY

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1. INTRODUCTION

In this paper, we study the growth of saturated powers of modules. In the case of an ideal I in a local ring (R, \mathfrak{m}) , the saturation of I^k in R is

$$(I^k)^{\text{sat}} = I^k :_R \mathfrak{m}^\infty = \bigcup_{n=1}^\infty I^k :_R \mathfrak{m}^n.$$

There are examples showing that the algebra of saturated powers of I , $\bigoplus_{k \geq 0} (I^k)^{\text{sat}}$ is not a finitely generated R -algebra; for instance, in many cases the saturated powers are the symbolic powers. As such, it cannot be expected that the “Hilbert function”, giving the length of the R -module $(I^k)^{\text{sat}}/I^k$, is very well behaved for large k . However, it can be shown that it is bounded above by a polynomial in k of degree d , where d is the dimension of R . We show that in many cases, there is a reasonable asymptotic behavior of this length.

Suppose that (R, \mathfrak{m}) is a Noetherian local domain of dimension $d \geq 1$. Let L be the quotient field of R . Let $\lambda(M)$ denote the length of an R -module M . Let F be a finitely generated free R -module, and let E be a submodule of F of rank e . Let $S = R[F] = \text{Sym}(F) = \bigoplus_{k \geq 0} F^k$ and let $R[E] = \bigoplus_{k \geq 0} E^k$ be the R -subalgebra of S generated by E . Let

$$E^k :_{F^k} \mathfrak{m}^\infty = \bigcup_{n=1}^\infty E^k :_{F^k} \mathfrak{m}^n$$

denote the saturation of E^k in F^k . We prove the following theorem:

Theorem 1.1. *Suppose that (R, \mathfrak{m}) is a local domain of depth ≥ 2 which is essentially of finite type over a field K of characteristic zero (or over a perfect field K such that R/\mathfrak{m} is algebraic over K). Let d be the dimension of R . Suppose that E is a rank e submodule of a finitely generated free R -module F . Then the limit*

$$(1) \quad \lim_{k \rightarrow \infty} \frac{\lambda(E^k :_{F^k} \mathfrak{m}^\infty / E^k)}{k^{d+e-1}} \in \mathbb{R}$$

exists.

The conclusions of this theorem follow from Theorem 3.2 and Remark 3.3.

Theorem 1.1 is proven in the case when $E = I$ is a homogeneous ideal and R is a standard graded normal K -algebra in our paper [3] with Hà, Srinivasan and Theodorescu. The theorem is proven with the additional assumptions that R is regular, $E = I$ is an ideal in $F = R$, and the singular locus of $\text{Spec}(R/I)$ is \mathfrak{m} in our paper [4] with Herzog and Srinivasan. Kleiman [13] has proven Theorem 1.1 in the case that E is a direct summand of F locally at every nonmaximal prime of R . The theorem is proven for E of low analytic deviation in [4], for the case of ideals, and by Ulrich and Validashti [19] for the case of modules; in the case of low analytic deviation, the limit is always zero. A generalization of

this problem to the case of saturations with respect to non \mathfrak{m} -primary ideals is investigated by Herzog, Puthenpurakal and Verma in [10]; they show that an appropriate limit exists for monomial ideals.

An example in [3] shows that even in the case when E is an ideal I in a regular local ring R , the limit may be irrational.

An important technique in the proof of Theorem 1.1 is to use a theorem of Lazarsfeld [14] showing that the volume of a line bundle on a complex projective variety can be expressed as a limit of numbers of global sections of powers of the line bundle; Lazarsfeld's theorem is deduced from an approximation theorem of Fujita [6] (generalizations of Fujita's result to positive characteristic are given in [17] and [15]).

We can interpret our results in terms of local cohomology. Let $F_L^k = F^k \otimes_R L$, where L is the quotient field of R , so that we have natural embeddings $E^k \subset F^k \subset F_L^k$ for all k . We have identities

$$H_{\mathfrak{m}}^0(F^k/E^k) \cong E^k :_{F^k} \mathfrak{m}^\infty / E^k \text{ and } H_{\mathfrak{m}}^1(E^k) \cong E^k :_{F_L^k} \mathfrak{m}^\infty / E^k.$$

Further, these two modules are equal if R has depth ≥ 2 .

We thus obtain the following corollary to Theorem 1.1, which shows that the epsilon multiplicity $\varepsilon(E)$ of a module, defined as a limsup in [19], actually exists as a limit.

Corollary 1.2. *Suppose that (R, \mathfrak{m}) is a local domain of depth ≥ 2 which is essentially of finite type over a field K of characteristic zero (or over a perfect field K such that R/\mathfrak{m} is algebraic over K). Let d be the dimension of R . Suppose that E is a rank e submodule of a finitely generated free R -module F . Then the limit*

$$\lim_{k \rightarrow \infty} \frac{(d+e-1)!}{k^{d+e-1}} \lambda(H_{\mathfrak{m}}^0(F^k/E^k)) \in \mathbb{R}$$

exists. Thus the epsilon multiplicity $\varepsilon(E)$ of E exists as a limit.

By the above identities of local cohomology, we see that (1) is equivalent to the statement that

$$(2) \quad \lim_{k \rightarrow \infty} \frac{H_{\mathfrak{m}}^0(F^k/E^k)}{k^{d+e-1}} = \lim_{k \rightarrow \infty} \frac{H_{\mathfrak{m}}^1(E^k)}{k^{d+e-1}} \in \mathbb{R}$$

exists when $\text{depth}(R) \geq 2$.

In Section 4, we extend our results to domains of dimension $d \geq 2$. We prove the following extension of Theorem 1.1, which shows that the second limit of (2),

$$\lim_{k \rightarrow \infty} \frac{H_{\mathfrak{m}}^1(E^k)}{k^{d+e-1}} \in \mathbb{R}$$

exists when R is a domain of dimension $d \geq 2$.

Theorem 1.3. *Suppose that (R, \mathfrak{m}) is a local domain of dimension $d \geq 2$ which is essentially of finite type over a field K of characteristic zero (or over a perfect field K such that R/\mathfrak{m} is algebraic over K). Suppose that E is a rank e submodule of a finitely generated free R -module F . Then the limit*

$$(3) \quad \lim_{k \rightarrow \infty} \frac{\lambda(E^k :_{F_L^k} \mathfrak{m}^\infty / E^k)}{k^{d+e-1}} \in \mathbb{R}$$

exists.

Theorem 1.3 follows from Theorem 4.1 and equations (24) and (6). We prove that the first limit of (2),

$$\lim_{k \rightarrow \infty} \frac{H_{\mathfrak{m}}^0(F^k/E^k)}{k^{d+e-1}} \in \mathbb{R}$$

exists when R is a domain of dimension $d \geq 2$ and E is embedded in F of rank $< d + e$. I thank Craig Huneke, Bernd Ulrich and Javid Validashti for pointing out this interesting consequence of Theorem 1.3.

Corollary 1.4. *Suppose that (R, \mathfrak{m}) is a local domain of dimension $d \geq 2$ which is essentially of finite type over a field K of characteristic zero (or over a perfect field K such that R/\mathfrak{m} is algebraic over K). Suppose that E is a rank e submodule of a finitely generated free R -module F . Suppose that $\gamma = \text{rank}(F) < d + e$. Then the limits*

$$(4) \quad \lim_{k \rightarrow \infty} \frac{\lambda(E^k :_{F^k} \mathfrak{m}^\infty / E^k)}{k^{d+e-1}} \in \mathbb{R}$$

and

$$(5) \quad \lim_{k \rightarrow \infty} \frac{(d+e-1)!}{k^{d+e-1}} \lambda(H_{\mathfrak{m}}^0(F^k/E^k)) \in \mathbb{R}$$

exist. In particular, the epsilon multiplicity $\varepsilon(E)$ of E exists as a limit.

In the case when $e = 1$ and $F = R$, we get the following statement.

Corollary 1.5. *Suppose that (R, \mathfrak{m}) is a local domain of dimension $d \geq 1$ which is essentially of finite type over a field K of characteristic zero (or over a perfect field K such that R/\mathfrak{m} is algebraic over K). Suppose that I is an ideal in R . Let $(I^k)^{\text{sat}} = I^k :_R \mathfrak{m}^\infty$ be the saturation of I^k . Then the limit*

$$\lim_{k \rightarrow \infty} \frac{\lambda((I^k)^{\text{sat}}/I^k)}{k^d} \in \mathbb{R}$$

exists.

Asymptotic polynomial like behavior of the length of extension functions is studied by Katz and Theodorescu [12], Theodorescu [18] and Crabbe, Katz, Striuli and Theodorescu [2]. By the local duality theorem, we obtain the following corollary to Theorem 1.1.

Corollary 1.6. *Suppose that (R, \mathfrak{m}) is a Gorenstein local domain of dimension $d \geq 2$ which is essentially of finite type over a field K of characteristic zero (or over a perfect field K such that R/\mathfrak{m} is algebraic over K). Suppose that E is a rank e submodule of a finitely generated free R -module F . Then the limit*

$$\lim_{k \rightarrow \infty} \frac{\lambda(\text{Ext}_R^d(F^k/E^k, R))}{k^{d+e-1}} \in \mathbb{R}$$

exists.

2. PRELIMINARIES

Suppose that (R, \mathfrak{m}) is a Noetherian local domain of dimension $d \geq 1$ with quotient field L . Let $\lambda_R(M)$ denote the length of an R -module M . When there is no danger of confusion, we will denote $\lambda_R(M)$ by $\lambda(M)$.

Let F be a finitely generated free R -module of rank γ , and let E be a submodule of F of rank e . Let $S = R[F] = \text{Sym}(F) = \bigoplus_{k \geq 0} F^k$, and let $R[E] = \bigoplus_{k \geq 0} E^k$ be the R -subalgebra of S generated by E . Let

$$E^k :_{F^k} \mathfrak{m}^\infty = \bigcup_{n=1}^\infty E^k :_{F^k} \mathfrak{m}^n$$

denote the saturation of E^k in F^k .

Let $F_L^k = F^k \otimes_R L$ (where L is the quotient field of R), so that we have natural embeddings $E^k \subset F^k \subset F_L^k$ for all k . Let $X = \text{Spec}(R)$, $\widetilde{E^k}$ be the sheafification of E on X and let u_1, \dots, u_s be generators of the ideal \mathfrak{m} .

There are identities

$$(6) \quad H^0(X \setminus \{\mathfrak{m}\}, \widetilde{E^k}) = \bigcap_{i=1}^s (E^k)_{u_i} = E^k :_{F_L^k} \mathfrak{m}^\infty.$$

From the exact sequence of cohomology groups

$$0 \rightarrow H_{\mathfrak{m}}^0(E^k) \rightarrow E^k \rightarrow H_{\mathfrak{m}}^0(X \setminus \{\mathfrak{m}\}, \widetilde{E^k}) \rightarrow H_{\mathfrak{m}}^1(E^k) \rightarrow 0,$$

we deduce that we have isomorphisms of R -modules

$$(7) \quad H_{\mathfrak{m}}^1(E^k) \cong E^k :_{F_L^k} \mathfrak{m}^\infty / E^k$$

for $k \geq 0$. The same calculation for F^k shows that

$$(8) \quad H_{\mathfrak{m}}^1(F^k) \cong F^k :_{F_L^k} \mathfrak{m}^\infty / F^k.$$

From the left exact local cohomology sequence

$$0 \rightarrow H_{\mathfrak{m}}^0(F^k/E^k) \rightarrow H_{\mathfrak{m}}^1(E^k) \rightarrow H_{\mathfrak{m}}^1(F^k),$$

we have that

$$(9) \quad H_{\mathfrak{m}}^0(F^k/E^k) \cong (E^k :_{F_L^k} \mathfrak{m}^\infty \cap F^k) / E^k = E^k :_{F^k} \mathfrak{m}^\infty / E^k.$$

From (6), and the fact that F^k is a free R -module, we have that $H^0(X \setminus \{\mathfrak{m}\}, \widetilde{F^k}) = F^k$ and

$$(10) \quad E^k :_{F_L^k} \mathfrak{m}^\infty = E^k :_{F^k} \mathfrak{m}^\infty \text{ if } R \text{ has depth } \geq 2.$$

Let ES be the ideal of S generated by E . We compute the degree n part of $(ES)^n$ from the formula

$$(11) \quad [(ES)^n]_n = E^n.$$

Let $R[\mathfrak{m}E] = \bigoplus_{n \geq 0} (\mathfrak{m}E)^n$ be the R -subalgebra of S generated by $\mathfrak{m}E$.

Let $X = \text{Spec}(R)$, $Y = \text{Proj}(R[\mathfrak{m}E])$ and $Z = \text{Proj}(R[E])$.

Write $R[E] = R[\bar{x}_1, \dots, \bar{x}_t]$ as a standard graded R -algebra, with $\deg \bar{x}_i = 1$ for all i . For $1 \leq i \leq t$, let

$$R_i = R[\frac{\bar{x}_1}{\bar{x}_i}, \dots, \frac{\bar{x}_t}{\bar{x}_i}],$$

and let $V_i = \text{Spec}(R_i)$ for $1 \leq i \leq t$. $\{V_i\}$ is an affine cover of Z . Let u_1, \dots, u_s be generators of the ideal \mathfrak{m} . For $1 \leq i \leq s$ and $1 \leq j \leq t$, let

$$R_{i,j} = R[\frac{u_\alpha \bar{x}_\beta}{u_i \bar{x}_j} \mid 1 \leq \alpha \leq s, 1 \leq \beta \leq t],$$

and $U_{i,j} = \text{Spec}(R_{i,j})$. Then $\{U_{i,j}\}$ is an affine cover of Y . Since

$$R_j[\frac{u_1}{u_i}, \dots, \frac{u_s}{u_i}] = R_{i,j},$$

we see that Y is the blow up of the ideal sheaf $\mathfrak{m}\mathcal{O}_Z$.

The structure morphism $f : Y \rightarrow X$ factors as a sequence of projective morphisms

$$Y \xrightarrow{g} Z \xrightarrow{h} X,$$

where Y is the blow up of the ideal sheaf $\mathfrak{m}\mathcal{O}_Z$. Define line bundles on Y by $\mathcal{L} = g^*\mathcal{O}_Z(1)$ and $\mathcal{M} = \mathfrak{m}\mathcal{O}_Y$. Then $\mathcal{O}_Y(1) \cong \mathcal{M} \otimes \mathcal{L}$.

We have $\mathcal{O}_Z(1)|_{V_j} = \bar{x}_j\mathcal{O}_{V_j}$, $\mathcal{L}|_{U_{i,j}} = \bar{x}_j\mathcal{O}_{U_{i,j}}$ and $\mathcal{M}|_{U_{i,j}} = u_i\mathcal{O}_{U_{i,j}}$.

We give three consequences (Proposition 2.1, Proposition 2.2 and Corollary 2.3) of Serre's fundamental theorem for projective morphisms which will be useful.

Proposition 2.1. $\bigoplus_{k \geq 0} H^i(Y, \mathcal{L}^k)$ are finitely generated $R[E]$ -modules for all $i \in \mathbb{N}$.

Proof. Let \widetilde{E}^k be the sheafification of E^k on X . From the natural surjections for $k \geq 0$ of \mathcal{O}_Z -modules $g^*(\widetilde{E}^k) \rightarrow \mathcal{O}_Z(k)$, we obtain surjections $f^*(\widetilde{E}^k) \rightarrow \mathcal{L}^k$ of \mathcal{O}_Y -modules, and a surjection $f^*(\bigoplus_{k \geq 0} \widetilde{E}^k) \rightarrow \bigoplus_{k \geq 0} \mathcal{L}^k$. Hence $\bigoplus_{k \geq 0} \mathcal{L}^k$ is a finitely generated $f^*(\bigoplus_{k \geq 0} \widetilde{E}^k)$ -module. By Theorem III.2.4.1 [8], $R^i f_*(\bigoplus_{k \geq 0} \mathcal{L}^k)$ is a finitely generated $\bigoplus_{k \geq 0} \widetilde{E}^k$ -module for $i \in \mathbb{N}$. Taking global sections on the affine X , we obtain the conclusions of the proposition. \square

Proposition 2.2. Suppose that A is a Noetherian ring, and $B = \bigoplus_{k \geq 0} B_k$ is a finitely generated graded A -algebra, which is generated by B_1 as an A -algebra. Let $C = \text{Spec}(A)$ and $D = \text{Proj}(B)$. Let $\alpha : D \rightarrow C$ be the structure morphism. Then there exists a positive integer \bar{k} such that $B_k = \Gamma(D, \mathcal{O}_D(k))$ for $k \geq \bar{k}$.

Proof. The ring $\bigoplus_{k \geq 0} \Gamma(D, \mathcal{O}_D(k))$ is a finitely generated graded B -module by Theorem III.2.4.1 [8]. Hence $(\bigoplus_{k \geq 0} \Gamma(D, \mathcal{O}_D(k)))/B$ is a finitely generated graded B -module. Since every element of this module is $B_+ = \bigoplus_{k > 0} B_k$ torsion, we have that $B_k/E_k = 0$ for $k \gg 0$. \square

Taking the maximum over the \bar{k} obtained from the above proposition applied to a finite affine cover of W , we obtain the following generalization of Proposition 2.2.

Corollary 2.3. Suppose that W is a Noetherian scheme and $\mathcal{B} = \bigoplus_{k \geq 0} \mathcal{B}_k$ is a finitely generated graded \mathcal{O}_W -algebra, which is locally generated by \mathcal{B}_1 as a \mathcal{O}_W -algebra. Let $W' = \text{Proj}(\mathcal{B})$ and let $\alpha : W' \rightarrow W$ be the structure morphism. Then there exists a positive integer \bar{k} such that $\mathcal{B}_k = \alpha_*\mathcal{O}_{W'}(k)$ for $k \geq \bar{k}$,

3. ASYMPTOTIC GROWTH

Proposition 3.1. Let (R, \mathfrak{m}) be a local domain of depth ≥ 2 . Let d be the dimension of R . Suppose that E is a rank e R -submodule of a finitely generated free R -module F . Let notation be as above. Then there exist positive integers k_0 , k_1 and τ such that

1) for $k \geq k_0$, $n \in \mathbb{Z}$ and $\mathfrak{p} \in X \setminus \{\mathfrak{m}\}$,

$$\Gamma(Y, \mathcal{M}^n \otimes \mathcal{L}^k)_{\mathfrak{p}} = (E^k)_{\mathfrak{p}}.$$

2) For $k \geq k_1$,

$$E^k :_{F^k} \mathfrak{m}^\infty = \Gamma(Y, \mathcal{M}^{-k\tau} \otimes \mathcal{L}^k).$$

Proof. We first establish 1). $U_i = \text{Spec}(R_{u_i})$ for $1 \leq i \leq s$ is an affine cover of $X \setminus \{\mathfrak{m}\}$. $g|_{f^{-1}(U_i)}$ is an isomorphism; in fact

$$f^{-1}(U_i) = \text{Proj}(R[\mathfrak{m}E]_{u_i}) = \text{Proj}(R[E]_{u_i}) = h^{-1}(U_i).$$

By Proposition 2.2, there exist positive integers a_i such that

$$\Gamma(f^{-1}(U_i), \mathcal{M}^{-n} \otimes \mathcal{L}^k) = \Gamma(h^{-1}(U_i), \mathcal{O}_Z(k)) = (E^k)_{u_i}$$

for $k \geq a_i$. Let $k_0 = \max\{a_1, \dots, a_s\}$. Then for $\mathfrak{p} \in U_i$

$$\Gamma(Y, \mathcal{M}^{-n} \otimes \mathcal{L}^k)_{\mathfrak{p}} = \Gamma(f^{-1}(U_i), \mathcal{M}^{-n} \otimes \mathcal{L}^k)_{\mathfrak{p}} = (E^k)_{\mathfrak{p}}$$

for $k \geq k_0$, establishing 1).

We now establish 2). Suppose that $n \geq 0$, and $k \geq 0$. Suppose that $\sigma \in E^k :_{F^k} \mathfrak{m}^n$. Let i, j be such that $1 \leq i \leq s$ and $1 \leq j \leq t$. $\sigma \mathfrak{m}^n \subset E^k$ implies $u_i^n \sigma \in E^k$ which implies there is an expansion

$$u_i^n \sigma = \sum_{n_1 + \dots + n_t = k} r_{n_1, \dots, n_t} \bar{x}_1^{n_1} \dots \bar{x}_t^{n_t}$$

with $r_{n_1, \dots, n_t} \in R$. Thus

$$u_i^n \sigma = \bar{x}_j^k \left(\sum_{n_1 + \dots + n_t = k} r_{n_1, \dots, n_t} \left(\frac{\bar{x}_1}{\bar{x}_j}\right)^{n_1} \dots \left(\frac{\bar{x}_t}{\bar{x}_j}\right)^{n_t} \right),$$

so that $\sigma \in u_i^{-n} \bar{x}_j^k R_{i,j}$. Thus

$$\sigma \in \cap_{i,j} u_i^{-n} \bar{x}_j^k R_{i,j} = \Gamma(Y, \mathcal{M}^{-n} \otimes \mathcal{L}^k).$$

We have established that for $k \geq 0$ and $n \geq 0$,

$$(12) \quad E^k :_{F^k} \mathfrak{m}^n \subset \Gamma(Y, \mathcal{M}^{-n} \otimes \mathcal{L}^k).$$

Recall that S is a polynomial ring $S = R[y_1, \dots, y_\gamma]$ over R , where γ is the rank of F . Let $W = \text{Proj}(S)$, with natural morphism $\alpha : W \rightarrow X$. Let \mathcal{I} be the sheafification of the graded ideal ES on W . We have expansions

$$\bar{x}_i = \sum_{l=1}^{\gamma} f_{il} y_l$$

with $f_{il} \in R$.

The inclusion $R[E] \subset S$ induces a rational map from W to Z .

Let $\beta : W' \rightarrow W$ be the blow up of the ideal sheaf \mathcal{I} . Let $\mathcal{N} = \mathcal{I}\mathcal{O}_{W'}$ be the induced line bundle. W' has an affine cover $A_{i,j} = \text{Spec}(T_{ij})$ for $1 \leq i \leq t$ and $1 \leq j \leq \gamma$ with

$$T_{ij} = R\left[\frac{y_1}{y_j}, \dots, \frac{y_\gamma}{y_j}\right]\left[\frac{\bar{x}_1}{\bar{x}_i}, \dots, \frac{\bar{x}_t}{\bar{x}_i}\right].$$

From the inclusions

$$R_i = R\left[\frac{\bar{x}_1}{\bar{x}_i}, \dots, \frac{\bar{x}_t}{\bar{x}_i}\right] \subset T_{ij}$$

we have induced morphisms $A_{i,j} \rightarrow V_i = \text{Spec}(R_i)$ which patch to give a morphism $\varphi : W' \rightarrow Z$ which is a resolution of indeterminacy of the rational map from W to Z .

We calculate for all i, j ,

$$\varphi^*(\mathcal{O}_Z(1))|_{A_{i,j}} = \bar{x}_i \mathcal{O}_{A_{i,j}} = y_j \left(\sum_l f_{il} \frac{y_l}{y_j} \right) \mathcal{O}_{A_{i,j}} = (\beta^* \mathcal{O}_W(1))|_{A_{i,j}},$$

to see that

$$(\beta^* \mathcal{O}_W(1)) \otimes \mathcal{N} \cong \varphi^* \mathcal{O}_Z(1).$$

By Corollary 2.3, there exists a positive integer $k_1 \geq k_0$ such that $\beta_* \mathcal{N}^k = \mathcal{I}^k$ for $k \geq k_1$. From the natural inclusion $\mathcal{O}_Z(k) \subset \varphi_* \varphi^* \mathcal{O}_Z(k)$, we have by the projection formula that for $k \geq k_1$,

$$(13) \quad \begin{aligned} h_* \mathcal{O}_Z(k) &\subset h_* \varphi_*(\varphi^* \mathcal{O}_Z(k)) = \alpha_* \beta_*(\beta^* \mathcal{O}_W(k) \otimes \mathcal{N}^k) \\ &= \alpha_*[\mathcal{O}_W(k) \otimes \beta_* \mathcal{N}^k] = \alpha_*[\mathcal{O}_W(k) \otimes \mathcal{I}^k] \\ &\subset \alpha_* \mathcal{O}_W(k) = \widetilde{F^k}, \end{aligned}$$

where $\widetilde{F^k}$ is the sheafification of the R -module F on X . Now we have

$$(14) \quad \begin{aligned} \Gamma(Y, \mathcal{M}^{-n} \otimes \mathcal{L}^k) &= \Gamma(X, f_*(\mathcal{M}^{-n} \otimes \mathcal{L}^k)) \\ &\subset \Gamma(X \setminus \{\mathfrak{m}\}, f_*(\mathcal{M}^{-n} \otimes \mathcal{L}^k)) = \Gamma(X \setminus \{\mathfrak{m}\}, h_* \mathcal{O}_Z(k)) \\ &\subset \Gamma(X \setminus \{\mathfrak{m}\}, \widetilde{F^k}) = F^k \end{aligned}$$

since R , and hence the free R -module F^k , have depth ≥ 2 .

From (11), we deduce that for $k, n \geq 0$,

$$(15) \quad \left((ES)^k :_S \mathfrak{m}^n S \right) \cap F^k = E^k :_{F^k} \mathfrak{m}^n.$$

By 1.5 [11] or Theorem 1.3 [16], there exists a positive integer τ such that

$$(ES)^k :_S \mathfrak{m}^{k\tau} S = (ES)^k :_S (\mathfrak{m} S)^\infty$$

for all $k \geq 0$. Thus from (15) we have that

$$(16) \quad E^k :_{F^k} \mathfrak{m}^{k\tau} = E^k :_{F^k} \mathfrak{m}^\infty$$

for $k \geq 0$. From (16), (12) and (14), we have inclusions

$$E^k :_{F^k} \mathfrak{m}^\infty \subset \Gamma(Y, \mathcal{M}^{-k\tau} \otimes \mathcal{L}^k) \subset F^k$$

for $k \geq k_1$. The conclusions of 2) of the proposition now follow from 1) of the proposition since $E^k :_{F^k} \mathfrak{m}^\infty$ is the largest R -submodule N of F^k which has the property that $N_{\mathfrak{p}} = (E^k)_{\mathfrak{p}}$ for $\mathfrak{p} \in X \setminus \{\mathfrak{m}\}$. □

Theorem 3.2. *Suppose that (R, \mathfrak{m}) is a local domain of depth ≥ 2 which is essentially of finite type over a field K of characteristic zero. Let d be the dimension of R . Suppose that E is a rank e submodule of a finitely generated free R -module F . Then the limit*

$$\lim_{k \rightarrow \infty} \frac{\lambda(E^k :_{F^k} \mathfrak{m}^\infty / E^k)}{k^{d+e-1}} \in \mathbb{R}$$

exists.

Proof. Let notation be as above.

First consider the short exact sequences

$$(17) \quad 0 \rightarrow \Gamma(Y, \mathcal{L}^k) / E^k \rightarrow E^k :_{F^k} \mathfrak{m}^\infty / E^k \rightarrow E^k :_{F^k} \mathfrak{m}^\infty / \Gamma(Y, \mathcal{L}^k) \rightarrow 0.$$

$\bigoplus_{k \geq 0} \Gamma(Y, \mathcal{L}^k)$ is a finitely generated $R[E]$ -module by Lemma 2.1. By 1) of Proposition 3.1, the support of the R -module $\Gamma(Y, \mathcal{L}^k) / E^k$ is contained in $\{\mathfrak{m}\}$ for all k . Since $(\bigoplus_{k \geq 0} \Gamma(Y, \mathcal{L}^k)) / R[E]$ is a finitely generated $R[E]$ -module, there is a positive integer r such that $\mathfrak{m}^r (\Gamma(Y, \mathcal{L}^k) / E^k) = 0$ for all k . Since $\dim R[E] / \mathfrak{m} R[E] \leq \dim R + \text{rank } E - 1 = d + e - 1$, and R / \mathfrak{m}^r is an Artin local ring, we have by the Hilbert-Serre theorem that $\lambda(\Gamma(Y, \mathcal{L}^k) / E^k)$ is a polynomial of degree less than or equal to $d + e - 2$ for $k \gg 0$. Thus

there exists a constant α such that $\lambda(\Gamma(Y, \mathcal{L}^k)/E^k) \leq \alpha k^{d+e-2}$ for all k . From (17), we are now reduced to showing that the limit

$$\lim_{k \rightarrow \infty} \frac{\lambda(E^k :_{F^k} \mathfrak{m}^\infty / \Gamma(Y, \mathcal{L}^k))}{k^{d+e-1}}$$

exists, from which we will have

$$(18) \quad \lim_{k \rightarrow \infty} \frac{\lambda(E^k :_{F^k} \mathfrak{m}^\infty / \Gamma(Y, \mathcal{L}^k))}{k^{d+e-1}} = \lim_{k \rightarrow \infty} \frac{\lambda(E^k :_{F^k} \mathfrak{m}^\infty / E^k)}{k^{d+e-1}}.$$

Taking global sections of the short exact sequences

$$0 \rightarrow \mathcal{L}^k \rightarrow \mathcal{M}^{-k\tau} \otimes \mathcal{L}^k \rightarrow \mathcal{M}^{-k\tau} \otimes \mathcal{L}^k \otimes (\mathcal{O}_Y / \mathfrak{m}^{k\tau} \mathcal{O}_Y) \rightarrow 0,$$

we obtain by Proposition 3.1 left exact sequences

$$(19) \quad 0 \rightarrow E^k :_{F^k} \mathfrak{m}^\infty / \Gamma(Y, \mathcal{L}^k) \rightarrow \Gamma(Y, \mathcal{M}^{-k\tau} \otimes \mathcal{L}^k \otimes (\mathcal{O}_Y / \mathfrak{m}^{k\tau} \mathcal{O}_Y)) \rightarrow H^1(Y, \mathcal{L}^k)$$

for $k \geq k_1$.

Let u_1, \dots, u_s be generators of the ideal \mathfrak{m} , and set $U_i = \text{Spec}(R_{u_i})$, so that $\{U_1, \dots, U_s\}$ is an affine cover of $X \setminus \{\mathfrak{m}\}$. Then $\mathcal{L}|_{f^{-1}(U_i)}$ is ample, so there exist positive integers b_i such that $R^1 f_*(\mathcal{L}^k) \otimes U_i = 0$ for $k \geq b_i$. Let $k_2 = \max\{b_1, \dots, b_s\}$. We have that the support of $H^1(Y, \mathcal{L}^k)$ is contained in $\{\mathfrak{m}\}$ for $k \geq k_2$.

$\bigoplus_{k \geq 0} H^1(Y, \mathcal{L}^k)$ is a finitely generated $R[E]$ -module by Lemma 2.1. Hence the submodule $M = \bigoplus_{k \geq k_2} H^1(Y, \mathcal{L}^k)$ is a finitely generated graded $R[E]$ -module. We have that $\mathfrak{m}^r M = 0$ for some positive integer r . Since

$$\dim R[E]/\mathfrak{m}R[E] \leq \dim R + \text{rank } E - 1 = d + e - 1,$$

and R/\mathfrak{m}^r is an Artin local ring, we have by the Hilbert-Serre theorem that $\lambda(H^1(Y, \mathcal{L}^k))$ is a polynomial of degree less than or equal to $d + e - 2$ for $k \gg 0$. Thus there exists a constant c such that

$$\lambda(H^1(Y, \mathcal{L}^k)) \leq ck^{d+e-2}$$

for all $k \geq 0$. By consideration of (18) and (19), we are reduced to proving that the limit

$$(20) \quad \lim_{k \rightarrow \infty} \frac{\lambda(H^0(Y, \mathcal{M}^{-k\tau} \otimes \mathcal{L}^k \otimes \mathcal{O}_Y / \mathfrak{m}^{k\tau} \mathcal{O}_Y))}{k^{d+e-1}}$$

exists.

If R/\mathfrak{m} is algebraic over K , let $K' = K$. If R/\mathfrak{m} is transcendental over K , let t_1, \dots, t_r be a lift of a transcendence basis of R/\mathfrak{m} over K to R . The rational function field $K(t_1, \dots, t_r)$ is contained in R . Let $K' = K(t_1, \dots, t_r)$. We have that R/\mathfrak{m} is finite algebraic over K' .

There exists an affine K' -variety $X' = \text{Spec}(A)$ such that R is the local ring of a closed point α of X' , and E extends to a submodule E' of A^γ , where γ is the rank of the free R -module F . We then have an inclusion of graded A -algebras $A[E'] \subset \text{Sym}(A^\gamma)$ which extends $R[E]$. Identify \mathfrak{m} with its extension to a maximal ideal of A . The structure morphism $Y' = \text{Proj}(A[\mathfrak{m}E']) \rightarrow X'$ is projective and its localization at \mathfrak{m} is $f : Y \rightarrow X$. Let \overline{X} be a projective closure of X' and let \tilde{Y} be a projective closure of Y' . X' is an open subset of \overline{X} and Y' is an open subset of \tilde{Y} . Let $\overline{Y} \rightarrow \tilde{Y}$ be the blow up of an ideal sheaf which gives a resolution of indeterminacy of the rational map from \tilde{Y} to \overline{X} . We may assume that the morphism $\overline{Y} \rightarrow \tilde{Y}$ is an isomorphism over the locus where the rational map is a morphism, and thus an isomorphism over the subset Y' of \tilde{Y} . Let $\overline{f} : \overline{Y} \rightarrow \overline{X}$ be the resulting morphism. We now establish that $\overline{f}^{-1}(X') = Y'$. Suppose that $p \in X'$ and $q \in \overline{f}^{-1}(p)$. Let V be a valuation ring of the function field L of \overline{Y} (which is also the

function field of Y') which dominates the local ring $\mathcal{O}_{\overline{Y},q}$. By assumption, V dominates the local ring $\mathcal{O}_{X',p}$. V dominates the local ring of a point on Y' , by the valuative criterion for properness (Theorem II.4.7 [9]) applied to the proper morphism $Y' \rightarrow X'$. Since V dominates the local ring of a unique point on \overline{Y} , we have that $q \in Y'$.

After possibly replacing \overline{Y} with the blow up of an ideal sheaf on \overline{Y} whose support is disjoint from Y' , we may assume that \mathcal{L} extends to a line bundle on \overline{Y} which we will also denote by \mathcal{L} . We will identify \mathfrak{m} with its extension to the ideal sheaf of the point α on \overline{X} , and identify \mathcal{M} with its extension $\mathfrak{m}\mathcal{O}_{\overline{Y}}$ to a line bundle on \overline{Y} . Let \mathcal{A} be an ample divisor on \overline{X} . Then there exists $l > 0$ such that $\mathcal{C} = \overline{f}^*(\mathcal{A}^l) \otimes \mathcal{L}$ is generated by global sections and is big.

Set $\mathcal{B} = \mathcal{C} \otimes \mathcal{M}^{-\tau}$. Tensor the short exact sequences

$$0 \rightarrow \mathcal{M}^{k\tau} \rightarrow \mathcal{O}_{\overline{Y}} \rightarrow \mathcal{O}_{\overline{Y}}/\mathfrak{m}^{k\tau}\mathcal{O}_{\overline{Y}} \cong \mathcal{O}_Y/\mathfrak{m}^{k\tau}\mathcal{O}_Y \rightarrow 0$$

with \mathcal{B}^k to obtain the short exact sequences

$$0 \rightarrow \mathcal{C}^k \rightarrow \mathcal{B}^k \rightarrow \mathcal{M}^{-k\tau} \otimes \mathcal{L}^k \otimes \mathcal{O}_Y/\mathfrak{m}^{k\tau}\mathcal{O}_Y \rightarrow 0$$

for $k \geq 0$. Taking global sections, we have exact sequences

$$(21) \quad 0 \rightarrow H^0(\overline{Y}, \mathcal{C}^k) \rightarrow H^0(\overline{Y}, \mathcal{B}^k) \rightarrow H^0(Y, \mathcal{M}^{-k\tau} \otimes \mathcal{L}^k \otimes \mathcal{O}_Y/\mathfrak{m}^{k\tau}\mathcal{O}_Y) \rightarrow H^1(\overline{Y}, \mathcal{C}^k).$$

For a coherent sheaf \mathcal{F} on \overline{Y} , let

$$h^i(\overline{Y}, \mathcal{F}) = \dim_{K'} H^i(\overline{Y}, \mathcal{F}).$$

Since \mathcal{C} is semiample (generated by global sections and big) and \overline{Y} has dimension $d+e-1$, we have that

$$\lim_{k \rightarrow \infty} \frac{h^1(\overline{Y}, \mathcal{C}^k)}{k^{d+e-1}} = 0.$$

This follows for instance from [5]. Since $\bigoplus_{k \geq 0} H^0(\overline{Y}, \mathcal{C}^k)$ is a finitely generated K' algebra of dimension $d+e$, as \mathcal{C} is generated by global sections and is big (or by the Riemann Roch theorem and the vanishing theorem of [5]) we have that the limit

$$\lim_{k \rightarrow \infty} \frac{h^0(\overline{Y}, \mathcal{C}^k)}{k^{d+e-1}} \in \mathbb{Q}$$

exists. Since \mathcal{B} is big, by the corollary to [6] given in Example 11.4.7 [14] or [3], we have that the limit

$$\lim_{k \rightarrow \infty} \frac{h^0(\overline{Y}, \mathcal{B}^k)}{k^{d+e-1}} \in \mathbb{R}$$

exists. From the sequence (21), we see that

$$\lim_{k \rightarrow \infty} \frac{h^0(Y, \mathcal{M}^{-k\tau} \otimes \mathcal{L}^k \otimes \mathcal{O}_Y/\mathfrak{m}^{k\tau}\mathcal{O}_Y)}{k^{d+e-1}} \in \mathbb{R}$$

exists. The conclusions of the theorem now follow from (20) and the formula

$$\begin{aligned} h^0(Y, \mathcal{M}^{-k\tau} \otimes \mathcal{L}^k \otimes \mathcal{O}_Y/\mathfrak{m}^{k\tau}\mathcal{O}_Y) &= \dim_{K'} H^0(Y, \mathcal{M}^{-k\tau} \otimes \mathcal{L}^k \otimes \mathcal{O}_Y/\mathfrak{m}^{k\tau}\mathcal{O}_Y) \\ &= [R/\mathfrak{m} : K'] \lambda(H^0(Y, \mathcal{M}^{-k\tau} \otimes \mathcal{L}^k \otimes \mathcal{O}_Y/\mathfrak{m}^{k\tau}\mathcal{O}_Y)). \end{aligned}$$

□

Remark 3.3. *The conclusions of Theorem 3.2 are also true if K is a perfect field of positive characteristic and R/\mathfrak{m} is algebraic over K . In this case we have that $K' = K$ in the proof of Theorem 3.2. Let \overline{K} be an algebraic closure of K . Since K is perfect, $\overline{Y} \times_K \overline{K}$ is reduced, and to compute the limit, we reduce to computing the sections of the*

pullback of \mathcal{B}^k on the disjoint union of the irreducible (integral) components of $\overline{Y} \times_K \overline{K}$. Fujita's approximation theorem is valid on varieties over an algebraically closed field of positive characteristic, as was shown by Takagi [17], from which the existence of the limit now follows.

Remark 3.4. Theorem 3.2 is proven for graded ideals in [3]. An example where the limit is an irrational number is given in [3]. The theorem is proven with the additional assumptions that R is regular, $E = I$ is an ideal in $F = R$, and the singular locus of $\text{Spec}(R/I)$ is \mathfrak{m} in [4]. Kleiman [13] has proven Theorem 3.2 in the case that E is a direct summand of F locally at every nonmaximal prime of R .

Corollary 3.5. Suppose that (R, \mathfrak{m}) is a local domain of depth ≥ 2 which is essentially of finite type over a field K of characteristic zero. Let d be the dimension of R . Suppose that E is a rank e submodule of a finitely generated free R -module F . Then the limit

$$\lim_{k \rightarrow \infty} \frac{(d + e - 1)!}{k^{d+e-1}} \lambda(H_{\mathfrak{m}}^0(F^k/E^k)) \in \mathbb{R}$$

exists. Thus the epsilon multiplicity $\varepsilon(E)$ of the module E , defined in [19] as a limsup, actually exists as a limit.

The example of [3] shows that $\varepsilon(E)$ may be an irrational number.

Proof. The corollary is immediate from Theorem 3.2 and (9). \square

Remark 3.6. The conclusions of Corollary 3.5 are valid if K is a perfect field of positive characteristic and R/\mathfrak{m} is algebraic over K , by Remark 3.3.

4. EXTENSION TO DOMAINS OF DIMENSION ≥ 2 .

In this section, we prove extensions of Theorem 1.1 and Corollary 1.2 to domains of dimension ≥ 2 . Let notation be as in Section 2.

Suppose that R is a domain of dimension $d \geq 2$ with a dualizing module. By the Theorem of Finiteness, Theorem VIII.2.1 (and footnote) [7],

$$(22) \quad \overline{R} = \Gamma(X \setminus \{\mathfrak{m}\}, \mathcal{O}_X) = \bigcap_{\mathfrak{p} \in X \setminus \{\mathfrak{m}\}} R_{\mathfrak{p}}$$

is a finitely generated R -module, which lies between R and its quotient field. Since \overline{R}/R is \mathfrak{m} -torsion,

$$(23) \quad \lambda_R(\overline{R}/R) < \infty.$$

Let $\mathfrak{m}_1, \dots, \mathfrak{m}_{\alpha}$ be the maximal ideals of \overline{R} which lie over \mathfrak{m} . By our construction,

$$0 = H_{\mathfrak{m}}^1(\overline{R}) = H_{\mathfrak{m}\overline{R}}^1(\overline{R}) = \bigoplus_{i=1}^{\alpha} H_{\mathfrak{m}_i\overline{R}}^1(\overline{R}),$$

so

$$H_{\mathfrak{m}_i\overline{R}_i}^1(\overline{R}_{\mathfrak{m}_i}) = H_{\mathfrak{m}_i\overline{R}}^1(\overline{R}) \otimes_{\overline{R}} \overline{R}_{\mathfrak{m}_i} = 0$$

for $1 \leq i \leq \alpha$, and thus $\text{depth}(\overline{R}_{\mathfrak{m}_i}) \geq 2$ for $1 \leq i \leq \alpha$.

Let $\overline{F} = F \otimes_R \overline{R}$ and $\overline{R}[\overline{F}] = \bigoplus_{k \geq 0} \overline{F}^k$, so that $\overline{F}^k \cong F^k \otimes_R \overline{R}$ for all k . Let $\overline{E} = \overline{R}E$ be the \overline{R} -submodule of \overline{F} generated by E . Let $\overline{R}[\overline{E}] = \bigoplus_{k \geq 0} \overline{E}^k$ be the \overline{R} -subalgebra of $\overline{R}[\overline{F}]$ generated by \overline{E} .

Let u_1, \dots, u_s be generators of the ideal \mathfrak{m} . For $k \in \mathbb{N}$, let \widetilde{E}^k be the sheafification of E^k on $X = \text{Spec}(R)$.

There are identities

$$(24) \quad H^0(X \setminus \{\mathfrak{m}\}, \widetilde{E^k}) = \cap_{i=1}^s (E^k)_{u_i} = E^k :_{\overline{F}^k} \mathfrak{m}^\infty.$$

From the exact sequence of cohomology groups

$$0 \rightarrow H_{\mathfrak{m}}^0(E^k) \rightarrow E^k \rightarrow H_{\mathfrak{m}}^0(X \setminus \{\mathfrak{m}\}, \widetilde{E^k}) \rightarrow H_{\mathfrak{m}}^1(E^k) \rightarrow 0,$$

we deduce that we have isomorphisms of R -modules

$$(25) \quad H_{\mathfrak{m}}^1(E^k) \cong E^k :_{\overline{F}^k} \mathfrak{m}^\infty / E^k$$

for $k \geq 0$. The same calculation for F^k shows that

$$(26) \quad H_{\mathfrak{m}}^1(F^k) \cong F^k :_{\overline{F}^k} \mathfrak{m}^\infty / F^k.$$

From the left exact local cohomology sequence

$$0 \rightarrow H_{\mathfrak{m}}^0(F^k/E^k) \rightarrow H_{\mathfrak{m}}^1(E^k) \rightarrow H_{\mathfrak{m}}^1(F^k),$$

we have that

$$(27) \quad H_{\mathfrak{m}}^0(F^k/E^k) \cong \left(E^k :_{\overline{F}^k} \mathfrak{m}^\infty \cap F^k \right) / E^k = E^k :_{F^k} \mathfrak{m}^\infty / E^k.$$

Theorem 4.1. *Suppose that (R, \mathfrak{m}) is a local domain of dimension $d \geq 2$ which is essentially of finite type over a field K of characteristic zero (or over a perfect field K such that R/\mathfrak{m} is algebraic over K). Suppose that E is a rank e submodule of a finitely generated free R -module F . Then the limit*

$$(28) \quad \lim_{k \rightarrow \infty} \frac{\lambda \left(E^k :_{\overline{F}^k} \mathfrak{m}^\infty / E^k \right)}{k^{d+e-1}} \in \mathbb{R}$$

exists.

Proof. Since $\overline{E}^k :_{\overline{F}^k} \mathfrak{m}^\infty / \overline{E}^k$ are finitely generated $\mathfrak{m}\overline{R}$ -torsion \overline{R} -modules, we have that

$$\overline{E}^k :_{\overline{F}^k} \mathfrak{m}^\infty / \overline{E}^k \cong \bigoplus_{i=1}^{\alpha} \left(\overline{E}_{\mathfrak{m}_i}^k :_{\overline{F}_{\mathfrak{m}_i}^k} \mathfrak{m}_i^\infty / \overline{E}_{\mathfrak{m}_i}^k \right).$$

By Theorem 1.1, we have that

$$\lim_{k \rightarrow \infty} \frac{\lambda_{\overline{R}_{\mathfrak{m}_i}} \left(\overline{E}_{\mathfrak{m}_i}^k :_{\overline{F}_{\mathfrak{m}_i}^k} \mathfrak{m}_i^\infty / \overline{E}_{\mathfrak{m}_i}^k \right)}{k^{d+e-1}}$$

exists for $1 \leq i \leq \alpha$. Since for any $\overline{R}_{\mathfrak{m}_i}$ module M we have that

$$\lambda_R(M) = [\overline{R}/\mathfrak{m}_i : R/\mathfrak{m}] \lambda_{\overline{R}_{\mathfrak{m}_i}}(M),$$

we conclude that

$$(29) \quad \lim_{k \rightarrow \infty} \frac{\lambda_R(\overline{E}^k :_{\overline{F}^k} \mathfrak{m}^\infty / \overline{E}^k)}{k^{d+e-1}}$$

exists. We have

$$\overline{E}^k :_{\overline{F}^k} \mathfrak{m}^\infty = \cap_{i=1}^s (\overline{E}^k)_{u_i} = \cap_{i=1}^s (E^k)_{u_i} = E^k :_{\overline{F}^k} \mathfrak{m}^\infty.$$

Consider the short exact sequences

$$(30) \quad 0 \rightarrow \overline{E}^k / E^k \rightarrow E^k :_{\overline{F}^k} \mathfrak{m}^\infty / E^k \rightarrow \overline{E}^k :_{\overline{F}^k} \mathfrak{m}^\infty / \overline{E}^k \rightarrow 0.$$

Now $\overline{R}[\overline{E}]/R[E]$ is a finitely generated $R[E]$ -module, and the support of the R -module \overline{E}^k/E^k is contained in $\{\mathfrak{m}\}$ for all k , so there exists a positive integer r such that \mathfrak{m}^r annihilates $\overline{R}[\overline{E}]/R[E]$. Thus (as in the argument following equation (17) in the proof of Theorem 3.2), we have that there exists a constant β such that

$$(31) \quad \lambda_R(\overline{E}^k/E^k) \leq \beta k^{d+e-2}$$

for all k . The conclusions of the proposition now follow from (29), (31) and (30). \square

I thank Craig Huneke, Bernd Ulrich and Javid Validashti for pointing out the following consequence of Theorem 4.1.

Corollary 4.2. *Suppose that (R, \mathfrak{m}) is a local domain of dimension $d \geq 2$ which is essentially of finite type over a field K of characteristic zero (or over a perfect field K such that R/\mathfrak{m} is algebraic over K). Suppose that E is a rank e submodule of a finitely generated free R -module F . Suppose that $\gamma = \text{rank}(F) < d + e$. Then the limits*

$$(32) \quad \lim_{k \rightarrow \infty} \frac{\lambda(E^k :_{F^k} \mathfrak{m}^\infty / E^k)}{k^{d+e-1}} \in \mathbb{R}$$

and

$$(33) \quad \lim_{k \rightarrow \infty} \frac{(d+e-1)!}{k^{d+e-1}} \lambda(H_{\mathfrak{m}}^0(F^k/E^k)) \in \mathbb{R}$$

exist. In particular, the epsilon multiplicity $\varepsilon(E)$ of E exists as a limit.

Proof. We will establish that the limit (32) exists. We have exact sequences

$$(34) \quad 0 \rightarrow E^k :_{F^k} \mathfrak{m}^\infty / E^k \rightarrow E^k :_{\overline{F}^k} \mathfrak{m}^\infty / E^k \rightarrow E^k :_{\overline{F}^k} \mathfrak{m}^\infty / E^k :_{F^k} \mathfrak{m}^\infty \rightarrow 0$$

and inclusions

$$(35) \quad E^k :_{\overline{F}^k} \mathfrak{m}^\infty / E^k :_{F^k} \mathfrak{m}^\infty = E^k :_{\overline{F}^k} \mathfrak{m}^\infty / \left((E^k :_{\overline{F}^k} \mathfrak{m}^\infty) \cap F^k \right) \rightarrow F^k :_{\overline{F}^k} \mathfrak{m}^\infty / F^k$$

for $k \geq 0$.

We have

$$(36) \quad F^k :_{\overline{F}^k} \mathfrak{m}^\infty / F^k = \overline{F}^k / F^k \cong (\overline{R}/R)^{\binom{k+\gamma-1}{\gamma-1}}.$$

Since $\gamma = \text{rank}(F) < d + e$, we have

$$\lim_{k \rightarrow \infty} \frac{\lambda_R(E^k :_{\overline{F}^k} \mathfrak{m}^\infty / E^k :_{F^k} \mathfrak{m}^\infty)}{k^{d+e-1}} = 0.$$

The existence of the limit (32) now follows from (34) and Theorem 4.1. The existence of the limit (33) is immediate from (32) and (27). \square

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